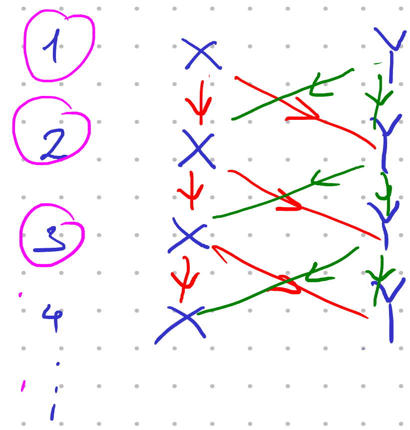
Zadatak 2.20.

Kutija X sadrži a belih i b crnih kuglica, kutija Y sadrži b belih i a crnih kuglica, pri čemu je $\boxed{a, b \gg 100.}$ Izvlače se kuglice jednaza drugom tako da ako se izvuče bela kuglica, sledeće izvlačenje je iz kutije X. Ako je prvo izvlačenje bilo iz kutije X, kolika je verovatnoća sa je n-ta kuglica bela za $n < 100$?

да

$X: \begin{cases} a & \text{белих} \\ b & \text{црних} \end{cases} \qquad Y: \begin{cases} b & \text{белих} \\ a & \text{црних} \end{cases}$



$$P_X = \frac{a}{a+b} \quad ; \quad P_Y = \frac{b}{a+b}$$

$$P[n] = \begin{cases} \boxed{X} & \text{извучена бела} \\ Y & \text{извучена бела} \end{cases} \Rightarrow$$

$$\Rightarrow \boxed{P[n] = P[n-1] \cdot P_X + (1 - P[n-1]) P_Y}$$

$$\Rightarrow P[n] = P[n-1] \cdot \frac{a}{a+b} + \frac{b}{a+b} - P[n-1]\frac{b}{a+b} = \frac{b}{a+b} + P[n-1] \cdot \frac{a-b}{a+b}$$

$$\boxed{P[n] - \frac{a-b}{a+b} P[n-1] = \frac{b}{a+b}}$$

Хомоген део. "$=\emptyset$"

$$P[n] - \frac{a-b}{a+b} P[n-1] = \emptyset$$

$$1 - \frac{a-b}{a+b} x^{-1} = \emptyset$$

$$\Rightarrow \boxed{x = \frac{a-b}{a+b}} \quad P_h = C\left(\frac{a-b}{a+b}\right)^n$$

$$P[n] = P_{hom} + P_{part.}$$

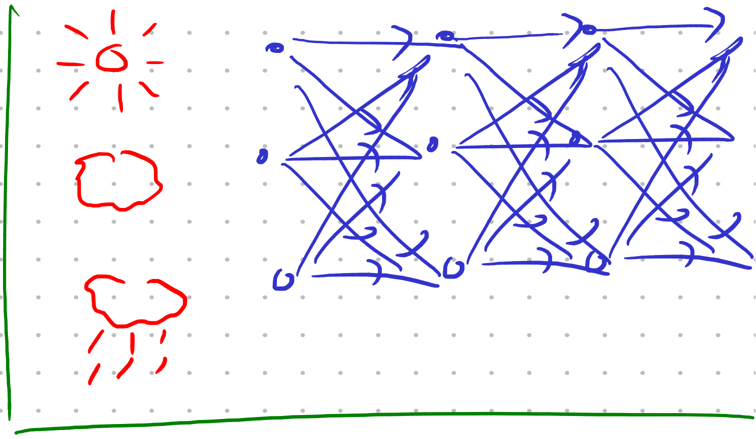
Партикуларни део: $P_p^{[n]} = P_\emptyset$

$$\Rightarrow P_\emptyset - \frac{a-b}{a+b} P_\emptyset = \frac{b}{a+b} \Rightarrow P_\emptyset \left(1 - \frac{a-b}{a+b}\right) = \frac{b}{a+b} \Rightarrow$$

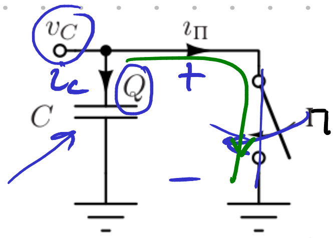
$$\Rightarrow P_\emptyset \frac{\cancel{a}+b-\cancel{a}+b}{\cancel{a+b}} = \frac{b}{\cancel{a+b}} \Rightarrow \boxed{P_\emptyset = \frac{b}{2b} = \frac{1}{2}}.$$

$$P[n] = \frac{1}{2} + C\left(\frac{a-b}{a+b}\right)^n \qquad P[\emptyset] = 1 \Rightarrow 1 = \frac{1}{2} + C$$

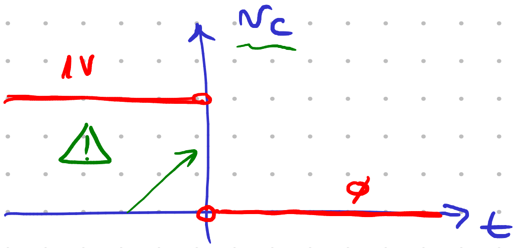
$$\Rightarrow \boxed{P[n] = \frac{1}{2}\left(1 + \left(\frac{a-b}{a+b}\right)^n\right)} \qquad \boxed{C = \frac{1}{2}}$$



2. У колу са слике познато је $C = 1\,\mu\text{F}$. Идеалан прекидач П је отворен, а кондензатор је оптерећен количином наелектрисања $Q = 1\,\mu\text{C}$. У тренутку $t_0 = 0$ затвара се прекидач. Одредити $v_C = v_C(t)$ и $i_\Pi = i_\Pi(t)$, за $-\infty < t < \infty$.



$v_c(t<0) = 1\,V, \quad v_c(t>0) = 0.$

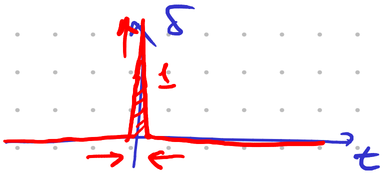


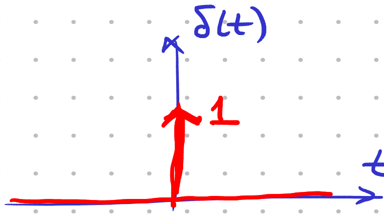
$i_c = C\boxed{\frac{dv_c}{dt}}$

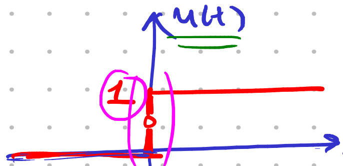
$C = \frac{Q}{v} \Rightarrow$

$\boxed{i_c = -Q\,\delta(t)}$ $\boxed{i_\Pi = Q\,\delta(t)}$ $\Rightarrow v = \frac{Q}{C}$

$$\delta(t) = \begin{cases} \delta(t \neq 0) = 0 \\ \boxed{\int_{-\infty}^{\infty} \delta(t)\,dt = 1} \end{cases}.$$



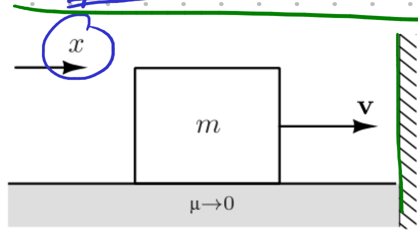


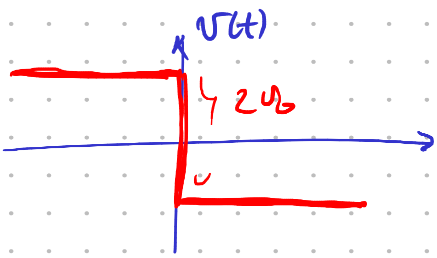


$v_c = 1V\,[1 - u(t)] \Rightarrow i_c = C\,\frac{dv_c}{dt} \Rightarrow$

$\Rightarrow i_c = C \cdot 1V \cdot \frac{d}{dt}(1 - u(t)) = 1\mu C \cdot (-\delta(t)) = -Q\,\delta(t) \rightarrow \boxed{\frac{du(t)}{dt} = \delta(t)}.$

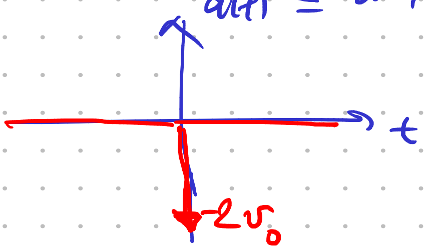
3. На слици је приказано круто тело масе m које може да се креће по подлози без трења. Брзина тела дата је као $\mathbf{v} = v(t)\,\mathbf{i}_x$. У тренутку $t_0 = 0$ блок се апсолутно еластично судара са непокретним зидом након чега се креће брзином $v(t) = -v_0$. (а) Одредити и изразити $v(t)$ за $-\infty < t < \infty$. (б) Одредити и нацртати временски дијаграм силе којом зид делује на блок $\mathbf{N} = N(t)\mathbf{i}_x$.





$\rightarrow$

$a(t) = dv/dt$



$\sim N$

5. Нацртати следеће континуалне сигнале:

$t=0$ $\quad$ $t=1$

(а) $x(t) = 2\mathrm{u}(t) - \mathrm{u}(t-1)$, и $\dfrac{\mathrm{d}x}{\mathrm{d}t}(t)$;

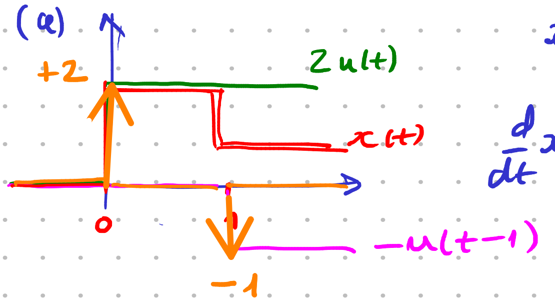
(б) $x(t) = \mathrm{u}(t+2) - 2\mathrm{u}(t) + \mathrm{u}(t-1)$, и $\dfrac{\mathrm{d}x}{\mathrm{d}t}(t)$;

(в) $x(t) = \cos(\pi t)[\delta(t+1) + \delta(t-1)]$, и $\boxed{\displaystyle\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau}$;

$\boxed{(г)\ \ x(t) = Ш_T(t) = \displaystyle\sum_{k=-\infty}^{n} \delta(t-kT),}$

где су $\mathrm{u}(t)$ и $\delta(t)$ јединична одскочна функција и Дираков импулс редом.

(a)

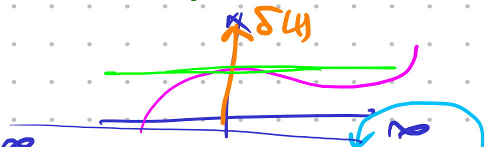


$x(t) = 2u(t) - u(t-1)$ $\qquad$ $\dfrac{d}{dt}$

$\dfrac{d}{dt}x(t) = 2\delta(t) - \delta(t-1).$

$\underbrace{f(t)}\cdot\delta(t) = f(\emptyset)\cdot\delta(t)$

f непрекидна око $t=\emptyset$



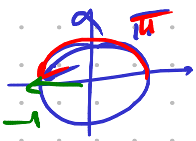
$\displaystyle\int_{-\infty}^{\infty} f(t)\cdot\delta(t)\,dt = \int_{-\infty} f(\emptyset)\,\delta(t)\,dt$

$\displaystyle = f(\emptyset)\cancel{\int_{-\infty}^{\infty}\delta(t)\,dt}^{\,=1} = f(\emptyset)$

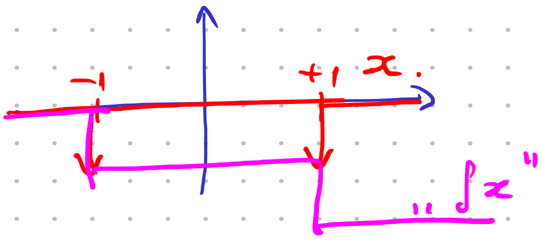
(в) $\underbrace{x(t) = \cos(\pi t)\left[\delta(t+1) + \delta(t-1)\right]} =$

$= \underbrace{\cos(\pi t)}_{\substack{t+1=0\\ @\,t=-1}}\delta(t+1) + \underbrace{\cos(\pi t)}_{\substack{t-1=0\\ @\,t=1}}\delta(t-1)$

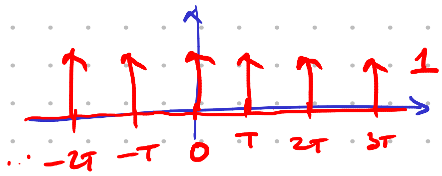
$= \cos(-\pi)\,\delta(t+1) + \cos(\pi)\,\delta(t-1)$



$= \underbrace{-1\,(\delta(t+1) + \delta(t-1))}$



(г) $Ш_T(t) = \displaystyle\sum_{k=-\infty}^{\infty} \delta(t-kT)$



comb_(t)

6. Нацртати следеће дискретне сигнале $x = x[n]$:

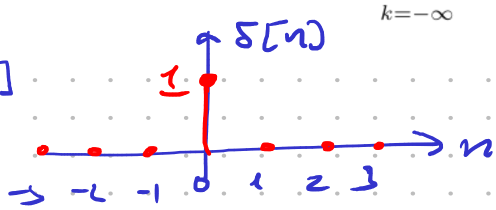
(а) $x[n] = \mathrm{u}[n] - 2\mathrm{u}[n-4]$, и $\boxed{y[n] = x[n] - x[n-1];}$

(б) $x[n] = n^2(\delta[n+2] - 2\delta[n-2])$, и $\displaystyle\sum_{k=-\infty}^{n} x[k]$

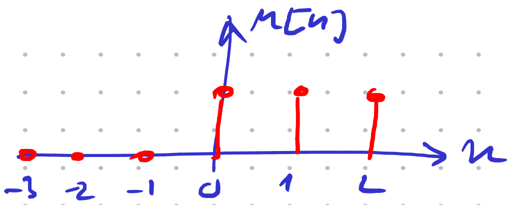
(в) $x[n] = (1-n)(\mathrm{u}[n+2] - \mathrm{u}[n-3])$

(г) $x[n] = \cos\dfrac{\pi n}{N}\left(\displaystyle\sum_{k=-\infty}^{\infty}\delta[n-kN]\right)\mathrm{u}[n]$, за $N=3$;

$\delta[n]$



$u[n]$



$\delta[n] = \begin{cases} 1, & n = \emptyset \\ \emptyset, & n \neq \emptyset \end{cases}$

$u[n] = \begin{cases} 1, & n \geq \emptyset \\ \emptyset, & n < \emptyset \end{cases}$

$\delta[n] = \{\ldots, \emptyset, \emptyset, 1, \emptyset, \emptyset, \ldots\}$

$$\delta[n] = u[n] - u[n-1] = \nabla u[n], \qquad \nabla x[n] = x[n] - x[n-1]$$

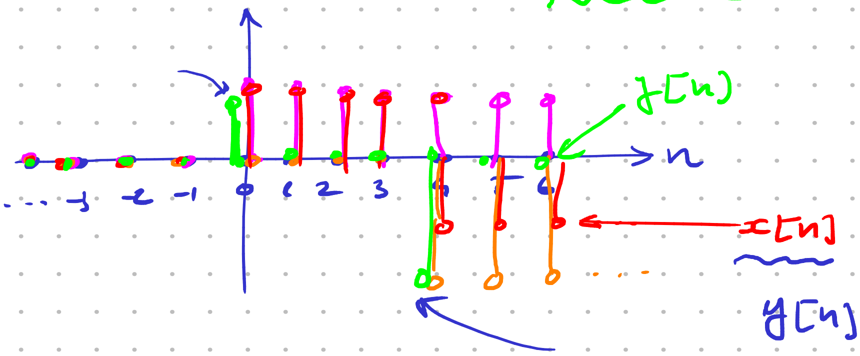
$$\Delta x[n] = x[n+1] - x[n]$$

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] = \sum \delta[n] \qquad \sum x[n] = \sum_{-\infty\,(m)}^{n} x[m]$$

$$\boxed{\sum \cdot \nabla = 1}$$

$$x[n] = \underbrace{\boxed{u[n]}}_{n=0} - \underbrace{\boxed{2u[n-4]}}_{n=4}, \qquad y[n] = \nabla x[n]$$

$$\boxed{\nabla \sim \frac{d}{dt}, \quad \sum \sim \int}$$



$$y[n] = \nabla\left(u[n] - 2u[n-4]\right)$$

$$\Rightarrow y[n] = \delta[n] - 2\delta[n-4]$$

$$y[n] = u[n] - u[n-1] - 2u[n-4] + 2u[n-5]$$

8. Одредити парну и непарну компоненту континуалних сигнала $x = x(t)$ за:

(а) $x(t) = e^{kt}$; (б) $x(t) = e^{j\omega_0 t}$; (в) $x(t) = \sin\left(\omega_0 t + \frac{\pi}{3}\right)$,

Паран $\quad x(t) = x(-t)$

Непаран: $\quad x(t) = -x(-t)$

$$x(t) = x_e(t) + x_o(t)$$

$$\left.\begin{array}{l} x_e(t) = x_e(-t) \\ x_o(t) = -x_o(-t) \end{array}\right\}$$

$$\left.\begin{array}{l} x(t) = x_e(t) + x_o(t) \\ x(-t) = \underbrace{x_e(-t)}_{x_e(t)} + \underbrace{x_o(-t)}_{-x_o(t)} \end{array}\right\}$$

$$x(t) = x_e(t) + x_o(t)$$

$$x(-t) = x_e(t) - x_o(t)$$

$$x_e(t) = \frac{x(t) + x(-t)}{2} \qquad x_o(t) = \frac{x(t) - x(-t)}{2}.$$

$$\boxed{\mathrm{Ev}\, x(t) = x_e(t)} \qquad \boxed{\mathrm{Od}\, x(t) = x_o(t)}$$

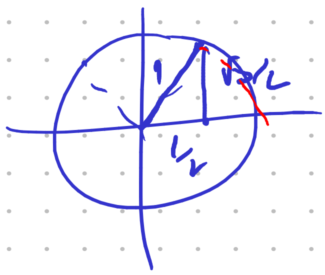
(а) $x(t) = e^{kt} \Rightarrow \mathrm{Ev}\, x(t) = \dfrac{e^{kt} + e^{-kt}}{2} = \cosh(kt).$

$$\mathrm{Od}\, x(t) = \frac{e^{kt} - e^{-kt}}{2} = \sinh(kt).$$

(б) $x(t) = e^{j\omega_0 t} \Rightarrow \mathrm{Ev}\, x(t) = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} = \cos(\omega_0 t)$

$$\mathrm{Od}\, x(t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{j\,2} \cdot j = j\sin(\omega_0 t).$$

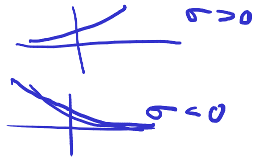
$$x(t) = \sin\left(\omega_0 t + \frac{\pi}{3}\right) = \underbrace{\cancel{\cos\frac{\pi}{3}}^{1/2} \sin\omega_0 t}_{Od} + \underbrace{\cancel{\sin\frac{\pi}{3}}^{\sqrt{3}/2} \cos\omega_0 t}_{Ev}$$



$$\int_{-a}^{a} x(t)\,dt = 2\int_0^a Ev\; x(t)\,dt$$

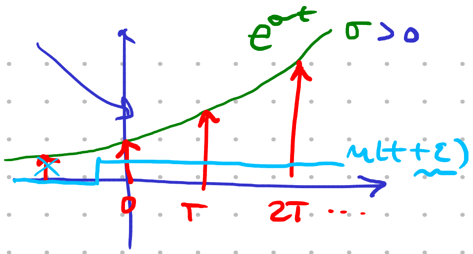
$$\frac{dx}{dt}(t=0) = \frac{d\,Odx}{dt}(0)$$

4. Нека је дат континуалан сигнал



$$x(t) = \underbrace{e^{\sigma t}\left(\overbrace{\sum_{k=-\infty}^{\infty}\delta(t-kT)}^{Ш_T(t)}\right)\boxed{u(t+\varepsilon)}} \quad (0<\varepsilon<T).$$

(а) Одредити услов које треба да задовољава параметар $\sigma \in \mathbb{R}$ тако да интеграл $\int_{-\infty}^{\infty} x(t)\,dt$ конвергира, и у том случају (б) израчунати тај интеграл.



$$x(t) = e^{\sigma t}\sum_{k=0}^{\infty}\delta(t-kT) \Rightarrow$$

$$\Rightarrow \int_{-\infty}^{\infty} x(t)\,dt = \int_{-\infty}^{\infty} e^{\sigma t}\sum_{k=0}^{\infty}\delta(t-kT)\,dt =$$

$u(t)\cdot\delta(t)$

$$\Rightarrow \int_{-\infty}^{\infty}\sum_{k=0}^{\infty} e^{\sigma}\,\delta(t-kT)\,dt = \sum_{k=0}^{\infty} \underbrace{\int_{-\infty} \cancel{e^{\sigma t}}^{e^{\sigma kT}}\,\underbrace{\delta(t-kT)}_{t=kT}dt}_{e^{\sigma kT}}$$

$$\Rightarrow \sum_{k=0}^{\infty}\left(e^{\sigma T}\right)^k \qquad |e^{\sigma T}| < 1 \qquad \sigma T < 0 \Rightarrow \boxed{\sigma < 0}$$

$$(б)\; \frac{1}{1-e^{\sigma T}}.$$